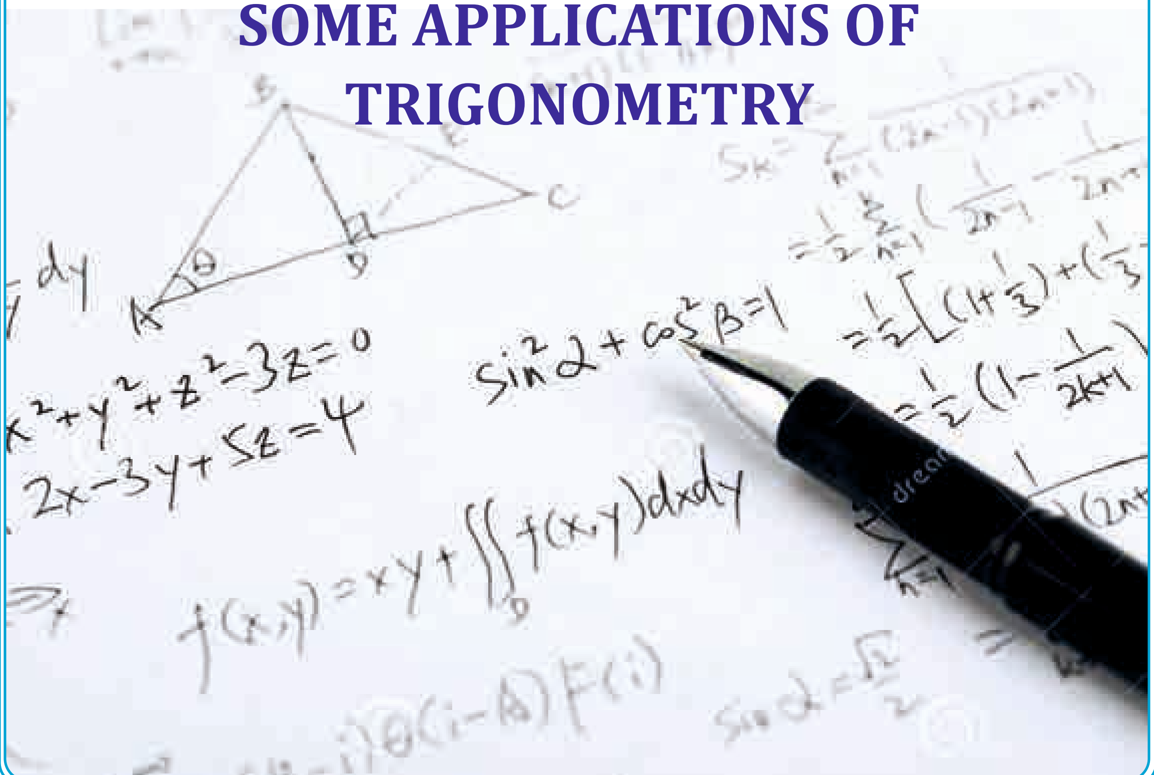
09

SOME APPLICATIONS OF TRIGONOMETRY



SOME APPLICATIONS OF TRIGONOMETRY

9

9.1 Introduction

In the previous chapter, you have studied about trigonometric ratios. In this chapter, you will be studying about some ways in which trigonometry is used in the life around you. Trigonometry is one of the most ancient subjects studied by scholars all over the world. As we have said in Chapter 8, trigonometry was invented because its need arose in astronomy. Since then the astronomers have used it, for instance, to calculate distances from the Earth to the planets and stars. Trigonometry is also used in geography and in navigation. The knowledge of trigonometry is used to construct maps, determine the position of an island in relation to the longitudes and latitudes.

Surveyors have used trigonometry for centuries. One such large surveying project of the nineteenth century was the **'Great Trigonometric Survey'** of British India for which the two largest-ever theodolites were built. During the survey in 1852, the highest mountain in the world was discovered. From a distance of over 160 km, the peak was observed from six different stations. In 1856, this peak was named after Sir George Everest, who had commissioned and first used the giant theodolites (see the figure alongside). The theodolites are now on display in the Museum of the Survey of India in Dehradun.

A Theodolite

(Surveying instrument, which is based on the Principles of trigonometry, is used for measuring angles with a rotating telescope)

In this chapter, we will see how trigonometry is used for finding the heights and distances of various objects, without actually measuring them.

9.2 Heights and Distances

Let us consider Fig. 8.1 of prvious chapter, which is redrawn below in Fig. 9.1.

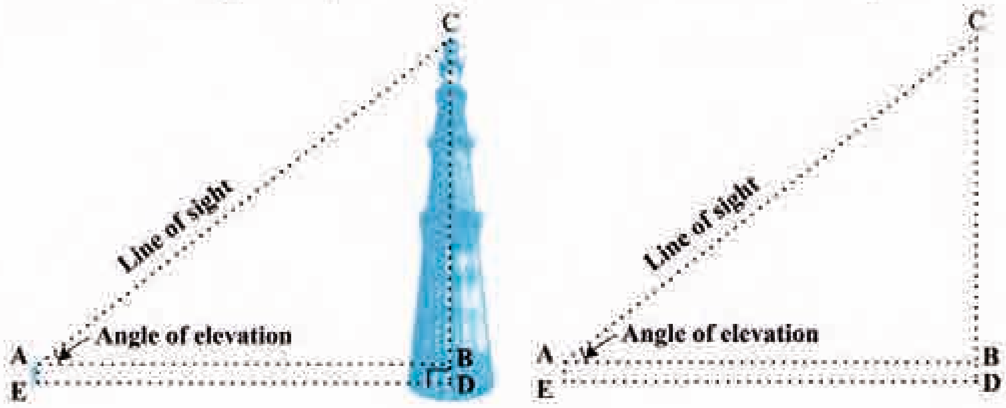


Fig. 9.1

In this figure, the line AC drawn from the eye of the student to the top of the minar is called the *line of sight*. The student is looking at the top of the minar. The angle BAC, so formed by the line of sight with the horizontal, is called the *angle of elevation* of the top of the minar from the eye of the student.

Thus, the **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer. The **angle of elevation** of the point viewed is the angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level, i.e., the case when we raise our head to look at the object (see Fig. 9.2).

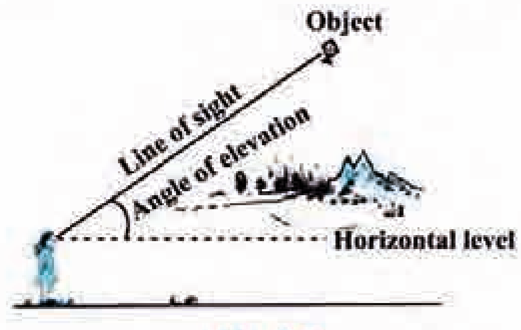


Fig. 9.2

Now, consider the situation given in Fig. 8.2. The girl sitting on the balcony is *looking down* at a flower pot placed on a stair of the temple. In this case, the line of sight is *below* the horizontal level. The angle so formed by the line of sight with the horizontal is called the *angle of depression*.

Thus, the **angle of depression** of a point on the object being viewed is the angle formed by the line of sight with the horizontal when the point is below the horizontal level, i.e., the case when we lower our head to look at the point being viewed (see Fig. 9.3).

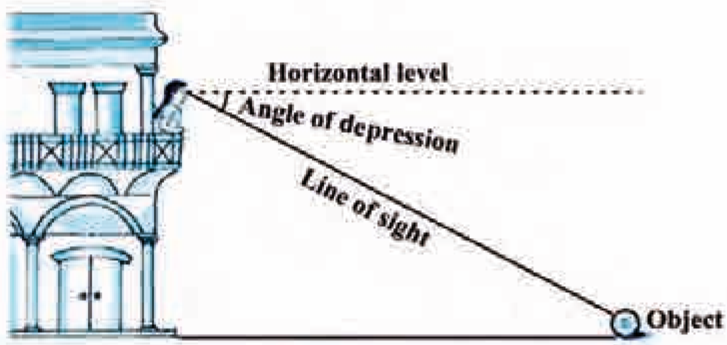


Fig. 9.3

Now, you may identify the lines of sight, and the angles so formed in Fig. 8.3. Are they angles of elevation or angles of depression?

Let us refer to Fig. 9.1 again. If you want to find the height CD of the minar without actually measuring it, what information do you need? You would need to know the following:

(i) the distance DE at which the student is standing from the foot of the minar

(ii) the angle of elevation, $\angle$ BAC, of the top of the minar

(iii) the height AE of the student.

Assuming that the above three conditions are known, how can we determine the height of the minar?

In the figure, CD = CB + BD. Here, BD = AE, which is the height of the student.

To find BC, we will use trigonometric ratios of $\angle$ BAC or $\angle$ A.

In Δ ABC, the side BC is the opposite side in relation to the known $\angle$ A. Now, which of the trigonometric ratios can we use? Which one of them has the two values that we have and the one we need to determine? Our search narrows down to using either tan A or cot A, as these ratios involve AB and BC.

Therefore, $\tan A = \frac{BC}{AB}$ or $\cot A = \frac{AB}{BC}$, which on solving would give us BC.

By adding AE to BC, you will get the height of the minar.

Now let us explain the process, we have just discussed, by solving some problems.

Example 1 : A tower stands vertically on the ground. From a point on the ground, which is 15 m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

Solution : First let us draw a simple diagram to represent the problem (see Fig. 9.4). Here AB represents the tower, CB is the distance of the point from the tower and $\angle$ ACB is the angle of elevation. We need to determine the height of the tower, i.e., AB. Also, ACB is a triangle, right-angled at B.

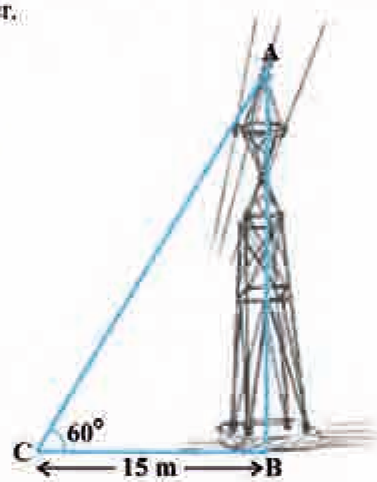


Fig. 9.4

To solve the problem, we choose the trigonometric ratio tan 60° (or cot 60°), as the ratio involves AB and BC.

Now, $\tan 60° = \frac{AB}{BC}$

i.e., $\sqrt{3} = \frac{AB}{15}$

i.e., $AB = 15\sqrt{3}$

Hence, the height of the tower is $15\sqrt{3}$ m.

Example 2 : An electrician has to repair an electric fault on a pole of height 5 m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work (see Fig. 9.5). What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder? (You may take $\sqrt{3} = 1.73$)

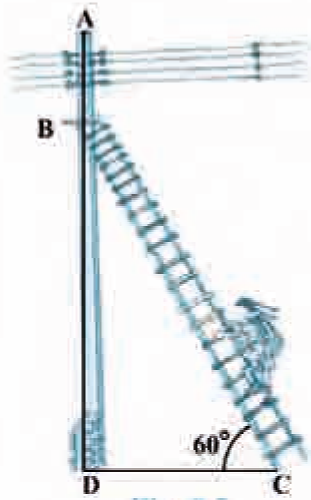


Fig. 9.5

Solution : In Fig. 9.5, the electrician is required to reach the point B on the pole AD.

So, BD = AD – AB = (5 – 1.3)m = 3.7 m.

Here, BC represents the ladder. We need to find its length, i.e., the hypotenuse of the right triangle BDC.

Now, can you think which trigonometic ratio should we consider?

It should be sin 60°.

So, $$\frac{BD}{BC} = \sin 60° \text{ or } \frac{3.7}{BC} = \frac{\sqrt{3}}{2}$$

Therefore, $$BC = \frac{3.7 \times 2}{\sqrt{3}} = 4.28 \text{ m (approx.)}$$

i.e., the length of the ladder should be 4.28 m.

Now, $$\frac{DC}{BD} = \cot 60° = \frac{1}{\sqrt{3}}$$

i.e., $$DC = \frac{3.7}{\sqrt{3}} = 2.14 \text{ m (approx.)}$$

Therefore, she should place the foot of the ladder at a distance of 2.14 m from the pole.

Example 3 : An observer 1.5 m tall is 28.5 m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

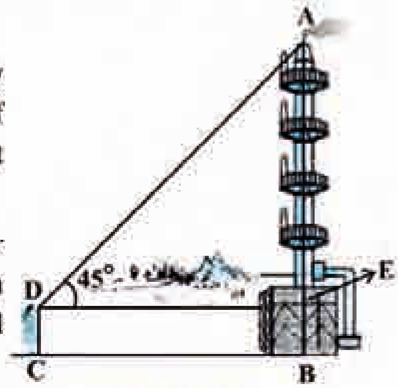


Fig. 9.6

Solution : Here, AB is the chimney, CD the observer and ∠ ADE the angle of elevation (see Fig. 9.6). In this case, ADE is a triangle, right-angled at E and we are required to find the height of the chimney.

We have AB = AE + BE = AE + 1.5

and DE = CB = 28.5 m

To determine AE, we choose a trigonometric ratio, which involves both AE and DE. Let us choose the tangent of the angle of elevation.

Now, $\tan 45° = \frac{AE}{DE}$

i.e., $1 = \frac{AE}{28.5}$

Therefore, AE = 28.5

So the height of the chimney (AB) = (28.5 + 1.5) m = 30 m.

Example 4 : From a point P on the ground the angle of elevation of the top of a 10 m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is 45°. Find the length of the flagstaff and the distance of the building from the point P. (You may take $\sqrt{3}$ = 1.732)

Solution : In Fig. 9.7, AB denotes the height of the building, BD the flagstaff and P the given point. Note that there are two right triangles PAB and PAD. We are required to find the length of the flagstaff, i.e., DB and the distance of the building from the point P, i.e., PA.

Since, we know the height of the building AB, we will first consider the right Δ PAB.

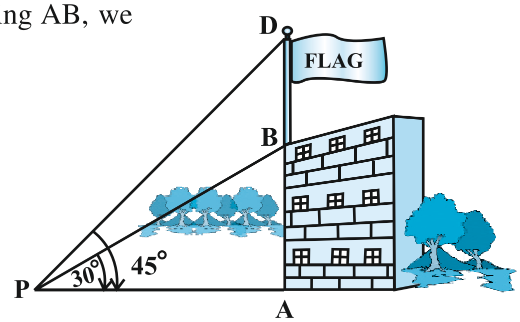


Fig. 9.7

We have $\tan 30° = \frac{AB}{AP}$

i.e., $\frac{1}{\sqrt{3}} = \frac{10}{AP}$

Therefore, $AP = 10\sqrt{3}$

i.e., the distance of the building from P is $10\sqrt{3}$ m = 17.32 m.

Next, let us suppose DB = x m. Then AD = $(10 + x)$ m.

Now, in right Δ PAD, $\tan 45° = \frac{AD}{AP} = \frac{10 + x}{10\sqrt{3}}$

Therefore, $1 = \frac{10 + x}{10\sqrt{3}}$

i.e., $x = 10\left(\sqrt{3}-1\right) = 7.32$

So, the length of the flagstaff is 7.32 m.

Example 5 : The shadow of a tower standing on a level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

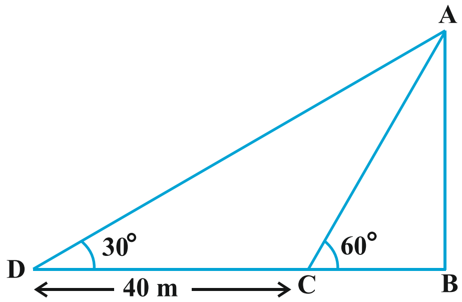


Fig. 9.8

Solution : In Fig. 9.8, AB is the tower and BC is the length of the shadow when the Sun's altitude is 60°, i.e., the angle of elevation of the top of the tower from the tip of the shadow is 60° and DB is the length of the shadow, when the angle of elevation is 30°.

Now, let AB be h m and BC be x m. According to the question, DB is 40 m longer than BC.

So, DB = $(40 + x)$ m

Now, we have two right triangles ABC and ABD.

In Δ ABC, $\tan 60° = \frac{AB}{BC}$

or, $\sqrt{3} = \frac{h}{x}$ (1)

In Δ ABD, $\tan 30° = \frac{AB}{BD}$

i.e., $\frac{1}{\sqrt{3}} = \frac{h}{x+40}$ (2)

From (1), we have $h = x\sqrt{3}$

Putting this value in (2), we get $\left(x\sqrt{3}\right)\sqrt{3} = x + 40$, i.e., $3x = x + 40$

i.e., $x = 20$

So, $h = 20\sqrt{3}$ [From (1)]

Therefore, the height of the tower is $20\sqrt{3}$ m.

Example 6 : The angles of depression of the top and the bottom of an 8 m tall building from the top of a multi-storeyed building are 30° and 45°, respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

Solution : In Fig. 9.9, PC denotes the multi-storyed building and AB denotes the 8 m tall building. We are interested to determine the height of the multi-storeyed building, i.e., PC and the distance between the two buildings, i.e., AC.

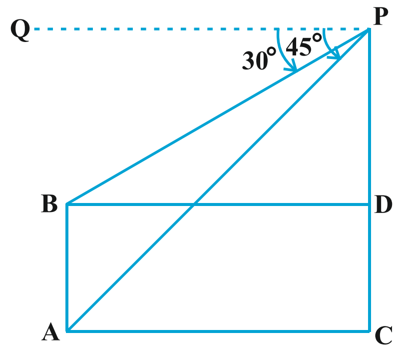


Fig. 9.9

Look at the figure carefully. Observe that PB is a transversal to the parallel lines PQ and BD. Therefore, ∠ QPB and ∠ PBD are alternate angles, and so are equal. So ∠ PBD = 30°. Similarly, ∠ PAC = 45°.

In right Δ PBD, we have

$$\frac{PD}{BD} = \tan 30° = \frac{1}{\sqrt{3}} \quad \text{or} \quad BD = PD\sqrt{3}$$

In right Δ PAC, we have

$$\frac{PC}{AC} = \tan 45° = 1$$

i.e., $PC = AC$

Also, $PC = PD + DC$, therefore, $PD + DC = AC$.

Since, AC = BD and DC = AB = 8 m, we get PD + 8 = BD = $PD\sqrt{3}$ (Why?)

This gives $$PD = \frac{8}{\sqrt{3}-1} = \frac{8\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = 4\left(\sqrt{3}+1\right)\text{m}.$$

So, the height of the multi-storeyed building is $\left\{4\left(\sqrt{3}+1\right)+8\right\}\text{m} = 4\left(3+\sqrt{3}\right)\text{m}$ and the distance between the two buildings is also $4\left(3+\sqrt{3}\right)\text{m}$.

Example 7 : From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are 30° and 45°, respectively. If the bridge is at a height of 3 m from the banks, find the width of the river.

Solution : In Fig 9.10, A and B represent points on the bank on opposite sides of the river, so that AB is the width of the river. P is a point on the bridge at a height of 3 m, i.e., DP = 3 m. We are interested to determine the width of the river, which is the length of the side AB of the Δ APB.

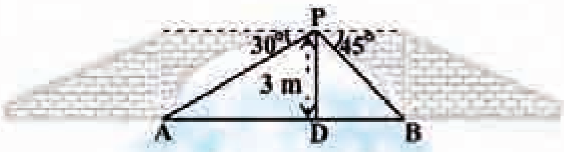


Fig. 9.10

Now, $AB = AD + DB$

In right Δ APD, $\angle A = 30°$.

So, $\tan 30° = \frac{PD}{AD}$

i.e., $\frac{1}{\sqrt{3}} = \frac{3}{AD}$ or $AD = 3\sqrt{3}$ m

Also, in right Δ PBD, $\angle B = 45°$. So, BD = PD = 3 m.

Now, $AB = BD + AD = 3 + 3\sqrt{3} = 3(1 + \sqrt{3})$ m.

Therefore, the width of the river is $3\left(\sqrt{3}+1\right)$m.

EXERCISE 9.1

1. A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30° (see Fig. 9.11).

Fig. 9.11

2. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.
3. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5 m, and

is inclined at an angle of 30° to the ground, whereas for elder children, she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?

4. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower, is 30°. Find the height of the tower.

5. A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

6. A 1.5 m tall boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

7. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.

8. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

9. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

10. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

11. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30° (see Fig. 9.12). Find the height of the tower and the width of the canal.

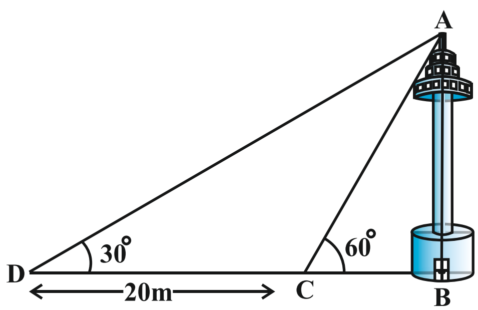


Fig. 9.12

12. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

13. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

14. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30° (see Fig. 9.13). Find the distance travelled by the balloon during the interval.

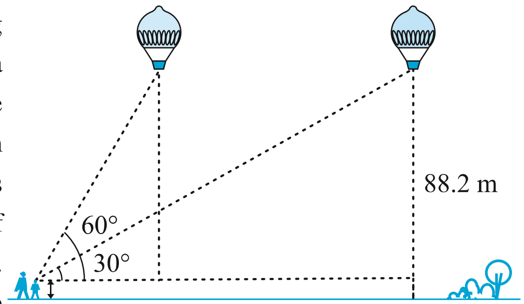


Fig. 9.13

15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

16. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

9.3 Summary

In this chapter, you have studied the following points :

1. (i) The **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer.

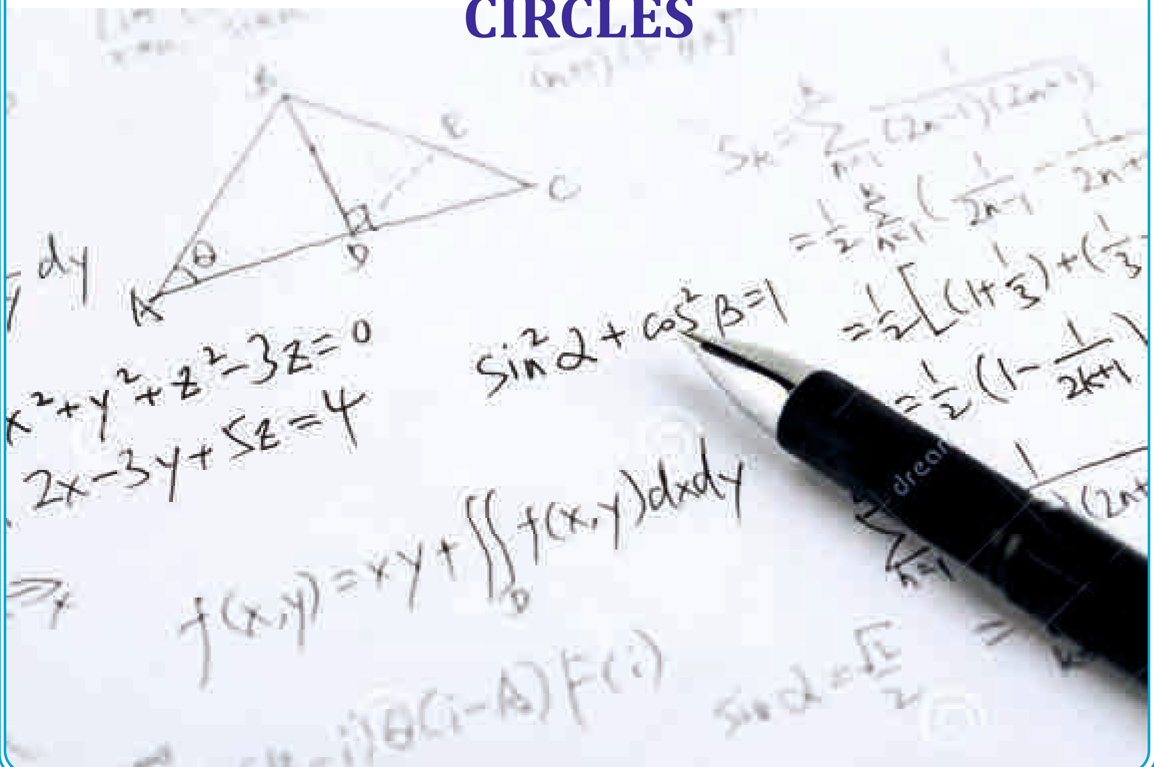
(ii) The **angle of elevation** of an object viewed, is the angle formed by the line of sight with the horizontal when it is above the horizontal level, i.e., the case when we raise our head to look at the object.

(iii) The **angle of depression** of an object viewed, is the angle formed by the line of sight with the horizontal when it is below the horizontal level, i.e., the case when we lower our head to look at the object.

2. The height or length of an object or the distance between two distant objects can be determined with the help of trigonometric ratios.

10

CIRCLES



CIRCLES 10

10.1 Introduction

You have studied in Class IX that a circle is a collection of all points in a plane which are at a constant distance (radius) from a fixed point (centre). You have also studied various terms related to a circle like chord, segment, sector, arc etc. Let us now examine the different situations that can arise when a circle and a line are given in a plane.

So, let us consider a circle and a line PQ. There can be three possibilities given in Fig. 10.1 below:

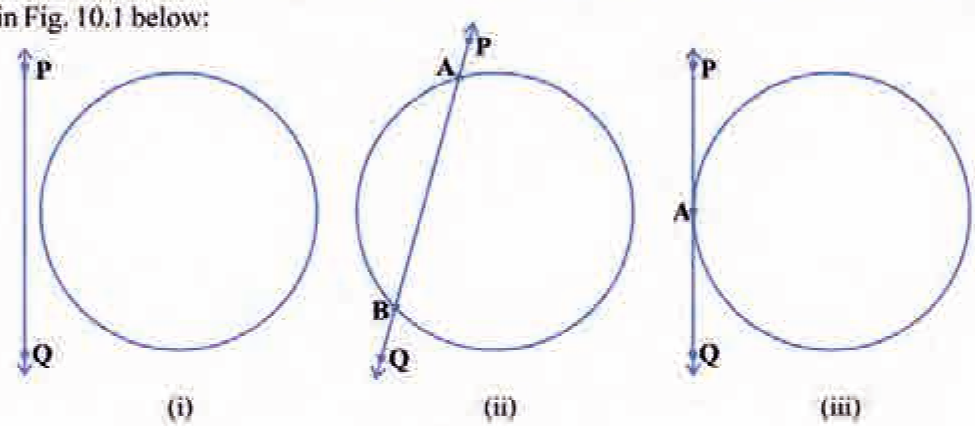


Fig. 10.1

In Fig. 10.1 (i), the line PQ and the circle have no common point. In this case, PQ is called a non-intersecting line with respect to the circle. In Fig. 10.1 (ii), there are two common points A and B that the line PQ and the circle have. In this case, we call the line PQ a secant of the circle. In Fig. 10.1 (iii), there is only one point A which is common to the line PQ and the circle. In this case, the line is called a tangent to the circle.

You might have seen a pulley fitted over a well which is used in taking out water from the well. Look at Fig. 10.2. Here the rope on both sides of the pulley, if considered as a ray, is like a tangent to the circle representing the pulley.

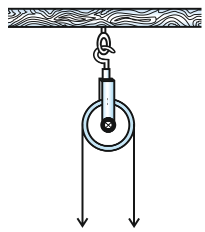


Fig. 10.2

Is there any position of the line with respect to the circle other than the types given above? You can see that there cannot be any other type of position of the line with respect to the circle. In this chapter, we will study about the existence of the tangents to a circle and also study some of their properties.

10.2 Tangent to a Circle

In the previous section, you have seen that **a tangent* to a circle is a line that intersects the circle at only one point**.

To understand the existence of the tangent to a circle at a point, let us perform the following activities:

Activity 1 : Take a circular wire and attach a straight wire AB at a point P of the circular wire so that it can rotate about the point P in a plane. Put the system on a table and gently rotate the wire AB about the point P to get different positions of the straight wire [see Fig. 10.3(i)].

In various positions, the wire intersects the circular wire at P and at another point Q_1 or Q_2 or Q_3, etc. In one position, you will see that it will intersect the circle at the point P only (see position A′B′ of AB). This shows that a tangent exists at the point P of the circle. On rotating further, you can observe that in all other positions of AB, it will intersect the circle at P and at another point, say R_1 or R_2 or R_3, etc. So, you can observe that **there is only one tangent at a point of the circle**.

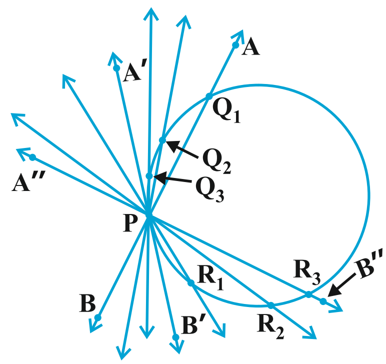


Fig. 10.3 (i)

While doing activity above, you must have observed that as the position AB moves towards the position A′ B′, the common point, say Q_1, of the line AB and the circle gradually comes nearer and nearer to the common point P. Ultimately, it coincides with the point P in the position A′B′ of A″B″. Again note, what happens if 'AB' is rotated rightwards about P? The common point R_3 gradually comes nearer and nearer to P and ultimately coincides with P. So, what we see is:

The tangent to a circle is a special case of the secant, when the two end points of its corresponding chord coincide.

*The word 'tangent' comes from the Latin word 'tangere', which means to touch and was introduced by the Danish mathematician Thomas Fineke in 1583.

Activity 2 : On a paper, draw a circle and a secant PQ of the circle. Draw various lines parallel to the secant on both sides of it. You will find that after some steps, the length of the chord cut by the lines will gradually decrease, i.e., the two points of intersection of the line and the circle are coming closer and closer [see Fig. 10.3(ii)]. In one case, it becomes zero on one side of the secant and in another case, it becomes zero on the other side of the secant. See the positions P′Q′ and P′′Q′′ of the secant in Fig. 10.3 (ii). These are the tangents to the circle parallel to the given secant PQ. This also helps you to see that there cannot be more than two tangents parallel to a given secant.

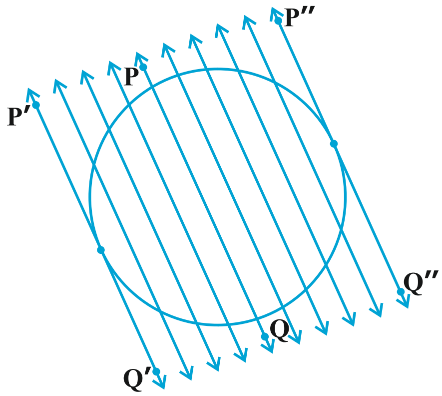


Fig. 10.3 (ii)

This activity also establishes, what you must have observed, while doing Activity 1, namely, a tangent is the secant when both of the end points of the corresponding chord coincide.

The common point of the tangent and the circle is called the **point of contact** [the point A in Fig. 10.1 (iii)]and the tangent is said to **touch** the circle at the common point.

Now look around you. Have you seen a bicycle or a cart moving? Look at its wheels. All the spokes of a wheel are along its radii. Now note the position of the wheel with respect to its movement on the ground. Do you see any tangent anywhere? (See Fig. 10.4). In fact, the wheel moves along a line which is a tangent to the circle representing the wheel. Also, notice that in all positions, the radius through the point of contact with the ground appears to be at right angles to the tangent (see Fig. 10.4). We shall now prove this property of the tangent.

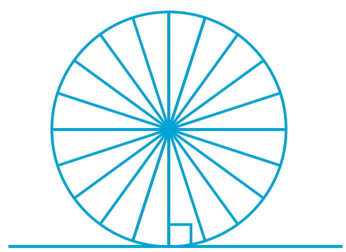


Fig. 10.4

Theorem 10.1 : *The tangent at any point of a circle is perpendicular to the radius through the point of contact.*

Proof : We are given a circle with centre O and a tangent XY to the circle at a point P. We need to prove that OP is perpendicular to XY.

Take a point Q on XY other than P and join OQ (see Fig. 10.5).

The point Q must lie outside the circle. (Why? Note that if Q lies inside the circle, XY will become a secant and not a tangent to the circle). Therefore, OQ is longer than the radius OP of the circle. That is,

$$OQ > OP.$$

Since this happens for every point on the line XY except the point P, OP is the shortest of all the distances of the point O to the points of XY. So OP is perpendicular to XY. (as shown in Theorem A1.7.)

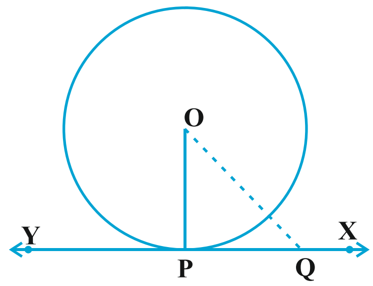


Fig. 10.5

Remarks :

1. By theorem above, we can also conclude that at any point on a circle there can be one and only one tangent.

2. The line containing the radius through the point of contact is also sometimes called the 'normal' to the circle at the point.

EXERCISE 10.1

1. How many tangents can a circle have?

2. Fill in the blanks :

(i) A tangent to a circle intersects it in ___________ point (s).

(ii) A line intersecting a circle in two points is called a ___________ .

(iii) A circle can have ___________ parallel tangents at the most.

(iv) The common point of a tangent to a circle and the circle is called ___________ .

3. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Length PQ is :

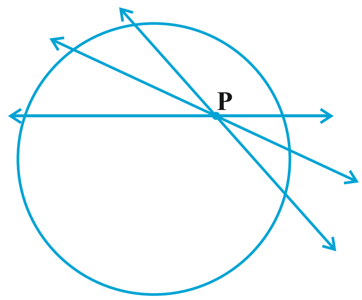
(A) 12 cm (B) 13 cm (C) 8.5 cm (D) $\sqrt{119}$ cm.

4. Draw a circle and two lines parallel to a given line such that one is a tangent and the other, a secant to the circle.

10.3 Number of Tangents from a Point on a Circle

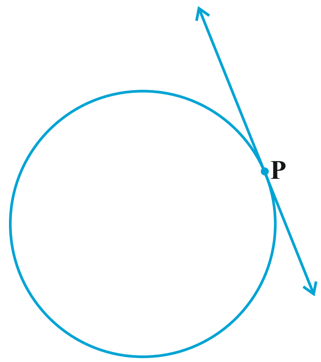
To get an idea of the number of tangents from a point on a circle, let us perform the following activity:

Activity 3 : Draw a circle on a paper. Take a point P inside it. Can you draw a tangent to the circle through this point? You will find that all the lines through this point intersect the circle in two points. So, it is not possible to draw any tangent to a circle through a point inside it [see Fig. 10.6 (i)].



(i)

Next take a point P on the circle and draw tangents through this point. You have already observed that there is only one tangent to the circle at such a point [see Fig. 10.6 (ii)].



(ii)

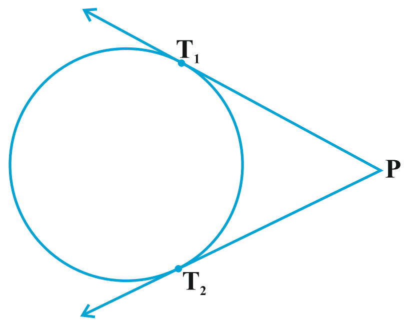
Finally, take a point P outside the circle and try to draw tangents to the circle from this point. What do you observe? You will find that you can draw exactly two tangents to the circle through this point [see Fig. 10.6 (iii)].

We can summarise these facts as follows:

Case 1 : There is no tangent to a circle passing through a point lying inside the circle.

Case 2 : There is one and only one tangent to a circle passing through a point lying on the circle.

Case 3 : There are exactly two tangents to a circle through a point lying outside the circle.



(iii)

Fig. 10.6

In Fig. 10.6 (iii), T_1 and T_2 are the points of contact of the tangents PT_1 and PT_2 respectively.

The length of the segment of the tangent from the external point P and the point of contact with the circle is called the **length of the tangent** from the point P to the circle.

Note that in Fig. 10.6 (iii), PT_1 and PT_2 are the lengths of the tangents from P to the circle. The lengths PT_1 and PT_2 have a common property. Can you find this? Measure PT_1 and PT_2. Are these equal? In fact, this is always so. Let us give a proof of this fact in the following theorem.

Theorem 10.2 : *The lengths of tangents drawn from an external point to a circle are equal.*

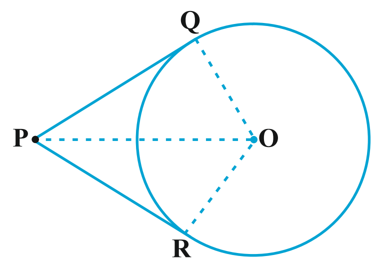


Fig. 10.7

Proof : We are given a circle with centre O, a point P lying outside the circle and two tangents PQ, PR on the circle from P (see Fig. 10.7). We are required to prove that PQ = PR.

For this, we join OP, OQ and OR. Then $\angle$ OQP and $\angle$ ORP are right angles, because these are angles between the radii and tangents, and according to Theorem 10.1 they are right angles. Now in right triangles OQP and ORP,

	OQ = OR	(Radii of the same circle)
	OP = OP	(Common)
Therefore,	Δ OQP $\cong$ Δ ORP	(RHS)
This gives	PQ = PR	(CPCT)

Remarks :

1. The theorem can also be proved by using the Pythagoras Theorem as follows:

$$PQ^2 = OP^2 - OQ^2 = OP^2 - OR^2 = PR^2 \text{ (As OQ = OR)}$$

which gives PQ = PR.

2. Note also that $\angle$ OPQ = $\angle$ OPR. Therefore, OP is the angle bisector of $\angle$ QPR, i.e., the centre lies on the bisector of the angle between the two tangents.

Let us take some examples.

Example 1 : Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

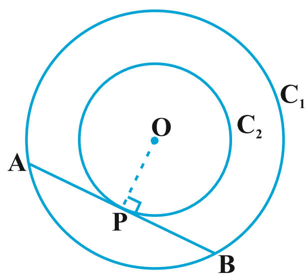


Fig. 10.8

Solution : We are given two concentric circles C_1 and C_2 with centre O and a chord AB of the larger circle C_1 which touches the smaller circle C_2 at the point P (see Fig. 10.8). We need to prove that AP = BP.

Let us join OP. Then, AB is a tangent to C_2 at P and OP is its radius. Therefore, by Theorem 10.1,

$$OP \perp AB$$

Now AB is a chord of the circle C_1 and OP ⊥ AB. Therefore, OP is the bisector of the chord AB, as the perpendicular from the centre bisects the chord,

i.e., $AP = BP$

Example 2 : Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that ∠ PTQ = 2 ∠ OPQ.

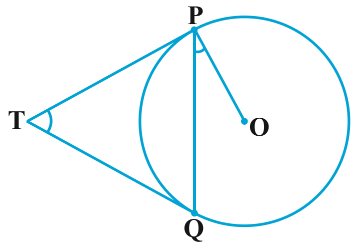


Fig. 10.9

Solution : We are given a circle with centre O, an external point T and two tangents TP and TQ to the circle, where P, Q are the points of contact (see Fig. 10.9). We need to prove that

$$\angle PTQ = 2 \angle OPQ$$

Let $\angle PTQ = \theta$

Now, by Theorem 10.2, TP = TQ. So, TPQ is an isosceles triangle.

Therefore, $\angle TPQ = \angle TQP = \frac{1}{2}(180° - \theta) = 90° - \frac{1}{2}\theta$

Also, by Theorem 10.1, $\angle OPT = 90°$

So, $\angle OPQ = \angle OPT - \angle TPQ = 90° - \left(90° - \frac{1}{2}\theta\right)$

$$= \frac{1}{2}\theta = \frac{1}{2}\angle PTQ$$

This gives $\angle PTQ = 2 \angle OPQ$

Example 3 : PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T (see Fig. 10.10). Find the length TP.

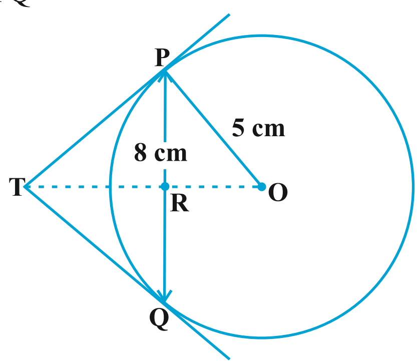


Fig. 10.10

Solution : Join OT. Let it intersect PQ at the point R. Then Δ TPQ is isosceles and TO is the angle bisector of ∠ PTQ. So, OT ⊥ PQ and therefore, OT bisects PQ which gives PR = RQ = 4 cm.

Also, $OR = \sqrt{OP^2 - PR^2} = \sqrt{5^2 - 4^2}$ cm $= 3$ cm.

Now, $\angle$ TPR + $\angle$ RPO = 90° = $\angle$ TPR + $\angle$ PTR (Why?)

So, $\angle$ RPO = $\angle$ PTR

Therefore, right triangle TRP is similar to the right triangle PRO by AA similarity.

This gives $\frac{TP}{PO} = \frac{RP}{RO}$, i.e., $\frac{TP}{5} = \frac{4}{3}$ or TP = $\frac{20}{3}$ cm.

Note : TP can also be found by using the Pythagoras Theorem, as follows:

Let TP = x and TR = y. Then

$$x^2 = y^2 + 16 \quad \text{(Taking right } \Delta \text{ PRT)} \quad (1)$$

$$x^2 + 5^2 = (y + 3)^2 \quad \text{(Taking right } \Delta \text{ OPT)} \quad (2)$$

Subtracting (1) from (2), we get

$$25 = 6y - 7 \quad \text{or} \quad y = \frac{32}{6} = \frac{16}{3}$$

Therefore, $$x^2 = \left(\frac{16}{3}\right)^2 + 16 = \frac{16}{9}(16+9) = \frac{16 \times 25}{9} \quad \text{[From (1)]}$$

or $$x = \frac{20}{3}$$

EXERCISE 10.2

In Q.1 to 3, choose the correct option and give justification.

1. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. The radius of the circle is
   (A) 7 cm (B) 12 cm
   (C) 15 cm (D) 24.5 cm

2. In Fig. 10.11, if TP and TQ are the two tangents to a circle with centre O so that $\angle$ POQ = 110°, then $\angle$ PTQ is equal to
   (A) 60° (B) 70°
   (C) 80° (D) 90°

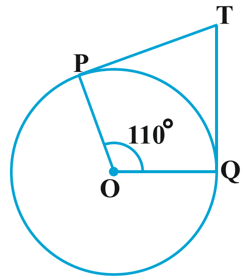


Fig. 10.11

3. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80°, then $\angle$ POA is equal to
   (A) 50° (B) 60°
   (C) 70° (D) 80°

4. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.

5. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

6. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

7. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

8. A quadrilateral ABCD is drawn to circumscribe a circle (see Fig. 10.12). Prove that

$$AB + CD = AD + BC$$

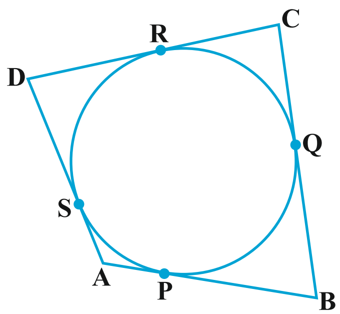


Fig. 10.12

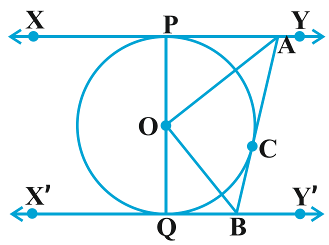


Fig. 10.13

9. In Fig. 10.13, XY and X′Y′ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and X′Y′ at B. Prove that $\angle AOB = 90°$.

10. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

11. Prove that the parallelogram circumscribing a circle is a rhombus.

12. A triangle ABC is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively (see Fig. 10.14). Find the sides AB and AC.

13. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

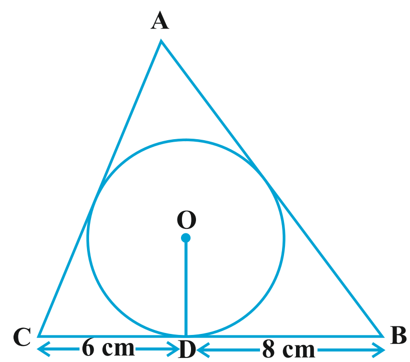


Fig. 10.14

10.4 Summary

In this chapter, you have studied the following points :

1. The meaning of a tangent to a circle.

2. The tangent to a circle is perpendicular to the radius through the point of contact.

3. The lengths of the two tangents from an external point to a circle are equal.